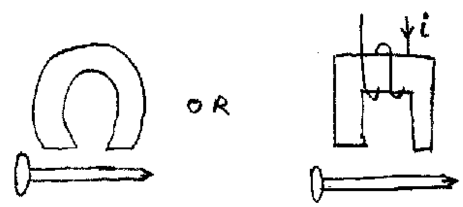
Chapter 9

Magnetic Devices II: Reluctance and Inductance

Up to now we have viewed magnetically produced forces as arising from the action of a magnetic field on moving charge carriers, e.g. the force on a current carrying conductor. Although this model gave the correct results for the slotted armature DC motor, in fact the force was actually acting on the iron of the armature, and not on the armature coils.

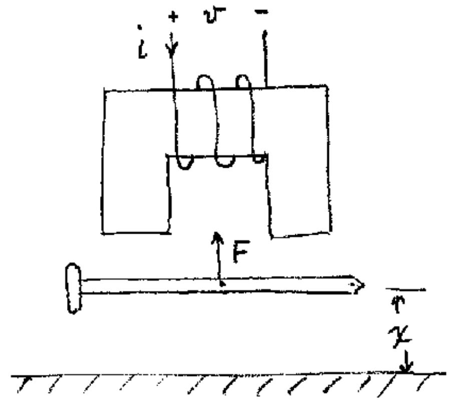
In some cases, there is no visible current on which the magnetic field can even *appear* to act. A familiar example of this is the force produced when picking up a nail with a magnet, either a permanent magnet or an electromagnet.



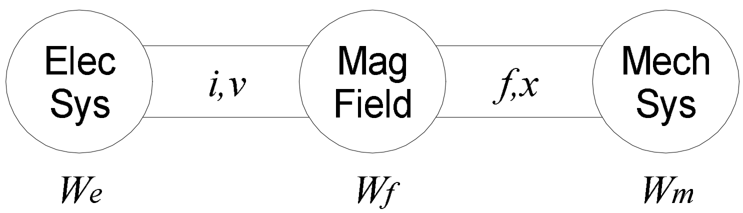
We could analyze this by looking at the interaction between the magnetic field of the magnet and the induced magnetic moment in the nail, but actually calculating this induced magnetism (which will vary with the distance from the magnet) would be very difficult. Instead, we will apply energy methods similar to those developed in Section 5.4.

9.1 Energy Balance in Magnetic Systems

If we take a closer look at an electromagnet picking up a nail, we find that work must be done to lift the weight of the nail through a distance x. This work is done by the magnetic field of the electromagnet whose energy in turn comes from the electrical source to which it is connected.



A block diagram of the process would look something like this:



where W_e is the total energy of the electrical system (e.g. a battery), W_m is the total energy of the mechanical system (the kinetic and gravitational potential energy of the nail), and W_f is the energy stored in the magnetic field.

If we assume a lossless system, then any energy flowing out of the electrical system must be accounted for by corresponding changes in the field and mechanical energies, i.e.:

$$dW_e = ei\,dt = dW_f + dW_m$$

or

$$dW_f = ei\,dt - dW_m$$

Using Faraday's law ($e = d\lambda/dt$) and (5.17)

$$dW_f = i\,d\lambda - F\,dx \tag{9.1}$$

We may express dW_f as

$$dW_f(\lambda, x) = \frac{\partial W_f}{\partial \lambda}\,d\lambda + \frac{\partial W_f}{\partial x}\,dx$$

so

$$i\,d\lambda - F\,dx = \frac{\partial W_f}{\partial \lambda}\,d\lambda + \frac{\partial W_f}{\partial x}\,dx$$

Since this must be true for all $d\lambda$ and dx:

$$i = \left.\frac{\partial W_f(\lambda, x)}{\partial \lambda}\right|_{x=const} \tag{9.2}$$

and

$$F = -\left.\frac{\partial W_f(\lambda, x)}{\partial x}\right|_{\lambda=const} \tag{9.3}$$

While it's easy to *postulate* holding λ constant while varying x, it's difficult to do so in practice. Constant λ requires that $\frac{d\lambda}{dt}$ be zero, but $\frac{d\lambda}{dt} = 0$ means that the voltage across the coil is zero, i.e. a short circuit. However, if $\lambda \neq 0$ there must be some current circulating in the coil, and if this current is not to decay away, we must have $R = 0$. This corresponds to holding Q constant in the electrostatic case where we place the desired charge on the plates and then leave them open circuited.

In practice it is often more useful to be able to express the force as a function of current. To this end we define the *coenergy* W_f':

$$W_f'(i, x) = i\lambda - W_f(\lambda, x) \tag{9.4}$$

Taking the differential

$$dW_f'(i, x) = d(i\lambda) - dW_f(\lambda, x)$$

Using

$$d(i\lambda) = i\, d\lambda + \lambda\, di$$

and (9.1) gives

$$\begin{aligned} dW_f' &= id\lambda + \lambda di - id\lambda + F dx \\ &= \lambda\, di + F\, dx \end{aligned}$$

We can express dW_f' as

$$dW_f' = \frac{\partial W_f'}{\partial i}\, di + \frac{\partial W_f'}{\partial x} dx$$

Using the same argument as above

$$\lambda = \left.\frac{\partial W_f'(i, x)}{\partial i}\right|_{x=const} \tag{9.5}$$

$$F = \left.\frac{\partial W_f'(i, x)}{\partial x}\right|_{i=const} \tag{9.6}$$

In a rotational system, $P = T\omega$ so we get

$$T = -\left.\frac{\partial W_f(\lambda, \theta)}{\partial \theta}\right|_{\lambda=const} \tag{9.7}$$

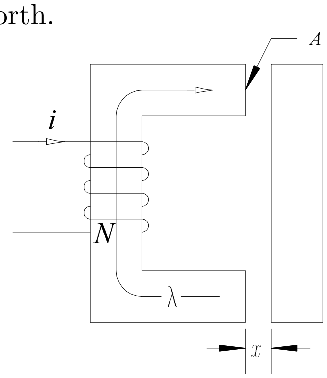
and

$$T = \left.\frac{\partial W_f'(i, \theta)}{\partial \theta}\right|_{i=const} \tag{9.8}$$

9.1.1 Finding W_f and W_f' in a Magnetic Circuit

In order for (9.3) or (9.6) to be useful, we must be able to find W_f and W_f', preferably in a form which can be easily differentiated. From (6.20) we know that this can be found in terms of the current i and the flux linkage λ, so our first step will be to determine how changes in x influence the relationship between these two variables.

Consider a magnetic circuit consisting of a coil and a two piece iron core with a pair of air gaps. The length of the gaps x may be varied by moving the "I" shaped section of the core back and forth.

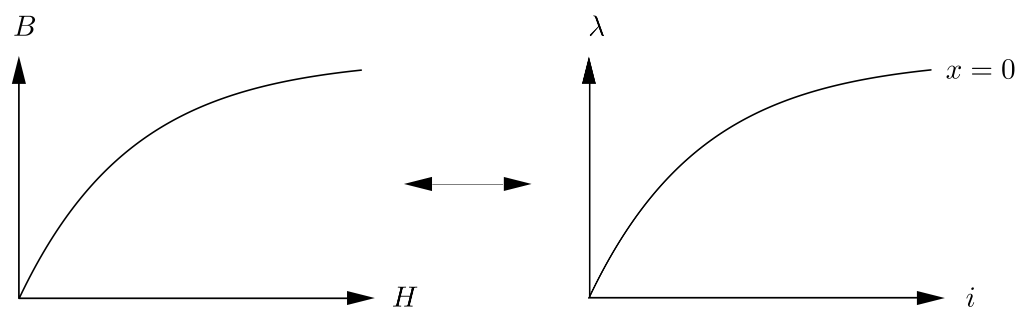


A = cross sectional area of core
l_c = length of flux path in core
l_g = length of flux path in gap = $2x$

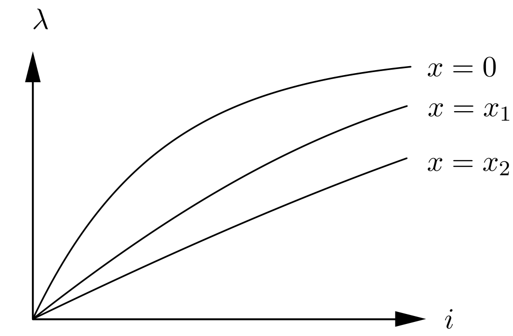
Because flux is continuous, if we neglect fringing,

$$B_c = B_g = B$$

The flux linkage is $\lambda = N\Phi = NAB$ and by Ampère's law $Ni = H_c l_c + H_g l_g$. For $x = 0$, $Ni = H_c l_c$ or $i = \frac{H_c l_c}{N}$. Since λ is proportional to B and i is proportional to H_c the plot of λ vs. i has the same shape as the B-H curve of the core material, but with with a different scale.

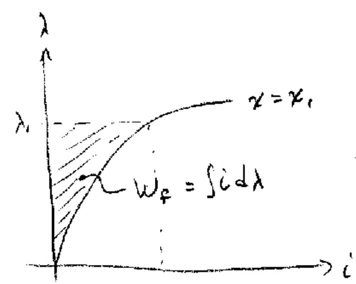


As x increases we get a family of curves that look like this:



9.1.1.1 Magnetic Energy

To determine the stored field energy, we can use the following procedure: We start with no stored energy, i.e. $W_f = 0$. If we assume that the core material has negligible hysteresis (i.e. no residual magnetization) then we must have $i = 0$. We force dW_m to be zero by fixing $x = x_1$, so that all electrical energy flowing into the system must be converted to field energy. As we increase the current from zero to i_1 the electrical power supplied to the system is $P = iv = i\frac{d\lambda}{dt}$. With x fixed, $dx = 0$, hence $dW_m = 0$ so



$$W_f = W_e = \int_0^{t_1} P\,dt = \int_0^{t_1} i\frac{d\lambda}{dt}\,dt = \int_0^{\lambda_1} i\,d\lambda = W_f(x, \lambda_1)$$

If we move to $x = x_2$ along a path of constant λ then

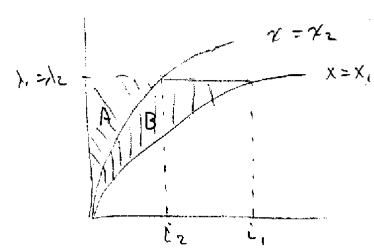
$$\begin{aligned} F dx \;=\; dW_m &= dW_e - dW_f \\ &= i d\lambda - dW_f \end{aligned}$$

For constant λ, $d\lambda = 0$ and

$$F = -\left.\frac{\partial W_f(\lambda, x)}{\partial x}\right|_{\lambda=const} \tag{9.9}$$

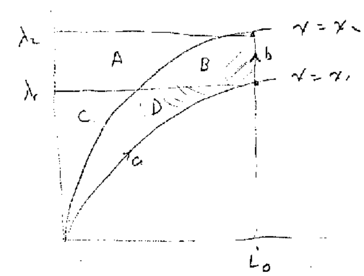
Graphically, W_f is equal to the area between the $\lambda - i$ curve and the λ axis. As x changes, the change in W_f will be the area between the two corresponding $\lambda - i$ curves. In the figure this is

$$dW_f = \mathrm{A} + \mathrm{B} - \mathrm{A} = \mathrm{B}$$



9.1.1.2 Magnetic Coenergy

Rather than holding λ constant while changing x as we did above, suppose instead we hold i constant. Starting with $x = x_1$ and $i = 0$, increase i to i_0 (path a in the figure). With i fixed at i_1, change x from x_1 to x_2 (path b). During this motion, the change in electrical energy is



$$\begin{aligned}\Delta W_e &= \int_{\lambda_1}^{\lambda_2} i_0 \, d\lambda \\ &= i_0(\lambda_2 - \lambda_1) \\ &= \text{area}(\text{A} + \text{B})\end{aligned}$$

Similarly, the change in field energy is $\Delta W_f = W_{f_2} - W_{f_1}$ where

$$\begin{aligned}W_{f_1} &= \text{area}(\text{C} + \text{D}) \\ W_{f_2} &= \text{area}(\text{C} + \text{A})\end{aligned}$$

or $\Delta W_f = \text{area}(\text{C} + \text{A} - \text{C} - \text{D}) = \text{area}(\text{A} - \text{D})$. Combining these we have

$$\begin{aligned}\Delta W_m &= \Delta W_e - \Delta W_f \\ &= \text{area}(\text{A} + \text{B}) - \text{area}(\text{A} - \text{D}) \\ &= \text{area}(\text{B} + \text{D})\end{aligned}$$

But this is just the increase in area *under* the $\lambda - i$ curve, i.e. in $\int \lambda \, di$. This integral also has units of energy, and since it is the complement of the stored field energy, it is called the *coenergy*:

$$W_f'(x, i_0) = \int_0^{i_0} \lambda(x, i) \, di \tag{9.10}$$

and

$$F = \left. \frac{\partial W_f'(x, i)}{\partial x} \right|_{i=const} \tag{9.11}$$

9.1.1.3 Alternate Forms for Energy and Coenergy

We can express the energy and coenergy in terms of other magnetic circuit variables. By substituting $\lambda = N\Phi$ and $\mathcal{F} = NI$ we get

$$W_f = \int i \, d\lambda = \int \frac{\mathcal{F}}{n} \, d(n\Phi) = \int \mathcal{F} \, d\Phi \tag{9.12}$$

and

$$W_f' = \int \Phi \, d\mathcal{F} \tag{9.13}$$

We could also integrate the magnetic energy density over the volume of the field:

$$W_f = \int_V w_m \, dv = \int_V \left(\int_0^{B_0} H \cdot dB \right) dv \tag{9.14}$$

In air this becomes

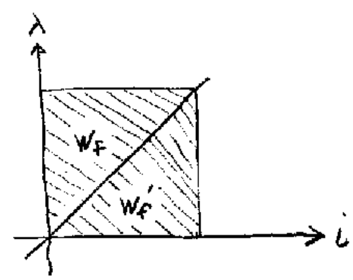
$$\begin{aligned} W_f &= \frac{1}{2} \int BH \, dv \\ &= \frac{1}{2} \int H^2 \mu_0 \, dv \\ &= \frac{1}{2} \int \frac{B^2}{\mu_0} \, dv \end{aligned}$$

Similarly, the coenergy is

$$W_f' = \int_V \left(\int_0^{H_0} B \cdot dH \right) dv \tag{9.15}$$

9.1.1.4 The Linear Case

For an ideal, linear magnetic material (or the linear region of a real material) we have $B = \mu H$ and $\lambda = Li$, where L is the inductance. We then get the following formulas for energy and coenergy:



	Energy W_f	Coenergy W_f'
Electric Circuit ($\frac{1}{2}\lambda i$)	$\frac{1}{2}\frac{\lambda^2}{L}$	$\frac{1}{2}Li^2$
Magnetic Circuit ($\frac{1}{2}\Phi\mathcal{F}$)	$\frac{1}{2}\frac{\Phi^2}{\mathcal{P}}$	$\frac{1}{2}\mathcal{P}\mathcal{F}^2$
Magnetic Field ($\frac{1}{2}BH$) (energy density)	$\frac{1}{2}\frac{B^2}{\mu}$	$\frac{1}{2}\mu H^2$

Table 9.1: Energy and Coenergy for Linear Magnetic Systems

In the constant λ case, $\frac{d\lambda}{dt} = 0$ and hence $v = 0$ and $dW_e = 0$. This means that all of the work done on the mechanical system must come from the magnetic field energy. I.e the force acts to *reduce* the field energy.

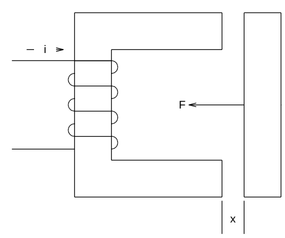
On the other hand, for i constant we have $F\,dx = dW_m = dW'_f$. In the linear case $W'_f = W_f$ so the force acts to *increase* the field energy. Since $dW_e = i\,d\lambda$ and $dW_f = \frac{1}{2}i\,d\lambda$, one half of the electrical input energy is converted to mechanical work and the other half goes to increase the field energy. This is called the *50-50 rule.*

9.2 Reluctance Force and Torque

We can now examine several prototype devices where the force (or torque) comes from the change in stored field energy due to changes in x (or θ). In these examples we will assume that the reluctance of the flux path through the iron is sufficiently smaller than that through the air gap that it may be ignored (i.e. that $\mu_r = \infty$ for the iron components).

9.2.1 Variable Gap Electromagnet

In the device on the right the flux in the gaps is



$$\Phi = \frac{\mathcal{F}}{\mathcal{R}} = \mathcal{F}\frac{\mu_0 A}{2x}$$

so

$$\mathcal{F} = \Phi\frac{2x}{\mu_0 A}$$

Using (9.12)

$$W_f = \int \mathcal{F}\,d\Phi = \int \Phi\frac{2x}{\mu_0 A}\,d\Phi = \frac{\Phi^2}{2}\frac{2x}{\mu_0 A}$$

If we hold the flux constant, the force is

$$F = -\frac{\partial W_f}{\partial x} = -\frac{\Phi^2}{\mu_0 A} = -\frac{\lambda^2}{n^2\mu_0 A}$$

For constant current (and therefore constant $\mathcal{F}$)

$$W'_f = \int \Phi\,d\mathcal{F} = \int \mathcal{F}\frac{\mu_0 A}{2x}\,d\mathcal{F} = \mathcal{F}^2\frac{\mu_0 A}{4x}$$

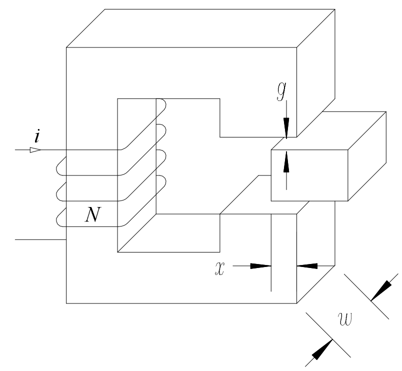
and

$$F = \frac{\partial W'_f}{\partial x} = -\frac{\mathcal{F}^2\mu_0 A}{4x^2} = -\frac{n^2 i^2\mu_0 A}{4x^2}$$

The negative sign for the force means it is in the direction of decreasing x, i.e. the force tends to close the gap.

9.2.2 Variable Overlap

If instead of having a gap of fixed area and variable length, we fix the length of the gap and allow its effective area to vary we get a significantly different relationship between force and displacement.



$$\begin{aligned}
\mathcal{R} &= \frac{2g}{\mu_0 A} = \frac{2g}{\mu_0 wx} \\
\Phi &= \frac{\mathcal{F}}{\mathcal{R}} = \mathcal{F}\frac{\mu_0 wx}{2g} \\
W'_f &= \int \Phi \, d\mathcal{F} = \frac{1}{2}\frac{\mu_0 wx}{2g}\mathcal{F}^2 \\
F &= \frac{\partial W'_f}{\partial x} = \frac{\mu_0 w}{4g}\mathcal{F}^2 = \frac{n^2 i^2 \mu_0 w}{4g}
\end{aligned}$$

The force still increases as i^2 but is now independent of x instead of increasing as $\frac{1}{x^2}$. This is similar to the difference in behavior of the parallel plate electrostatic actuator between perpendicular and transverse motion.

9.2.3 Rotary Reluctance Machine

It is also possible to build a rotary variable reluctance device where the gap area varies with angle. In this case

$$\mathcal{R} = \frac{l}{\mu A} = \frac{2g}{\mu_0 wr\theta}$$

where θ is the angle of overlap between the rotor and stator. The coenergy is

$$W'_f = \frac{1}{2}Li^2$$

where the inductance is

$$L = \frac{\lambda}{i} = \frac{N\Phi}{\mathcal{F}/N} = N^2\frac{\Phi}{\mathcal{F}} = \frac{N^2}{\mathcal{R}} = \frac{N^2\mu_0 wr\theta}{2g}$$

so the torque is

$$T = \frac{\partial W'_f}{\partial \theta} = \frac{1}{2}i^2\frac{dL}{d\theta} = \frac{N^2\mu_0 wr}{4g}i^2 \qquad (9.16)$$

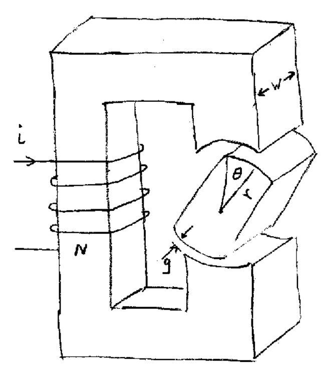
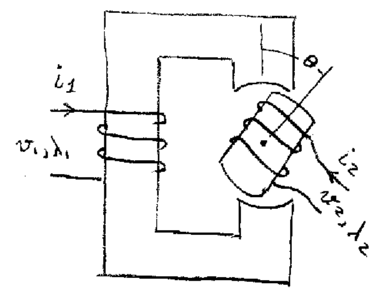


Figure 9.1: Rotary Reluctance Device

9.2.4 Multiply Excited Systems

So far we have only considered systems with a single electrical input, so called *singly excited* systems. We will also need to be able to analyze systems with multiple independent windings, or with a mixture of electrically excited coils and permanent magnets, i.e. *multiply excited systems.*

We can convert the system of Figure 9.1 to a doubly excited system by winding a second coil around the rotor. We can find the magnetic field energy using the same techinque we used previously of holding θ fixed and increasing the current from $i = 0$. However in this case, we must consider two separate currents, as well as the mutual flux that couples the windings of both coils.



In this case we have

$$dW_e = v_1 i_1 \, dt + v_2 i_2 \, dt$$

Since $v \, dt = d\lambda$, this becomes

$$dW_e = i_1 \, d\lambda_1 + i_2 \, d\lambda_2 \tag{9.17}$$

The total flux linking winding 1 is

$$\lambda_1 = L_{11} i_1 + L_{12} i_2 \tag{9.18}$$

where L_{11} is the *self inductance* of winding 1 and L_{12} is the *mutual inductance* between winding 2 and winding 1. Similarly

$$\lambda_2 = L_{21} i_1 + L_{22} i_2 \tag{9.19}$$

For simplicity, we can write $L_1 = L_{11}$, $L_2 = L_{22}$, and $M = L_{12} = L_{21}$. Substituting (9.18) and (9.19) into (9.17) we get

$$dW_e = L_1 i_1 \, di_1 + (M i_1 \, di_2 + M i_2 \, di_1) + L_2 i_2 \, di_2 = L_1 i_1 \, di_1 + M \, d(i_1 i_2) + L_2 i_2 \, di_2$$

Since θ is fixed $dW_m = 0$ and

$$W_f = W_e = \int dW_e = \frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2$$

For the linear case, $W_f' = W_f$ so

$$T = \frac{\partial W_f'}{\partial \theta} = \frac{1}{2} i_1^2 \frac{dL_1}{d\theta} + \frac{1}{2} i_2^2 \frac{dL_2}{d\theta} + i_1 i_2 \frac{dM}{d\theta} \tag{9.20}$$

The first two terms are reluctance torques, similar to that found in Section 9.2.3. Indeed, if we set $i_2 = 0$ (9.20) becomes (9.16), as we would expect. The third term represents the *alignment torque* between the magnetic fields of the rotor and stator. Alignment torque will be dealt with in more detail in the next chapter.

It is significant to note that variations in the stator self inductance can be eliminated by making the rotor fully cylindrical and smooth. This will eliminate the corresponding reluctance torque, so for example, the system on the right will produce no torque. Likewise, a smooth cylindrical stator structure will eliminate variations in rotor self inductance.